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ON CONFORMING MIXED FINITE ELEMENT METHODS FOR THE INHOMOGENEOUS STATIONARY NAVIER-STOKES EQUATIONS

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Abstract

We consider the stationary Navier-Stokes equations in the case where both the partial differential equations and boundary conditions are inhomogeneous. Under certain conditions on the data, we prove the existence and uniqueness of the solution of a weak formulation of the equations. Next, a conforming mixed finite element method is presented and optimal estimates for the error of the approximate solution are provided. In addition, the convergence properties of iterative methods for the solution of the discrete nonlinear algebraic systems resulting from the finite element algorithm are analyzed. Numerical examples, using an efficient choice of finite element spaces, are also provided.

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1. INTRODUCTION

The Navier-Stokes equations, which describe the motions of viscous incompressible fluids, have been the object of considerable research. Some studies have been directed at improving our understanding of various properties of the solutions of these equations, e.g. existence, uniqueness and regularity. Most of the available mathematical results concerning such properties are collected in [1], [2], [3] and [4]. Other studies have considered the approximate solution of the Navier-Stokes equations. Finite element methods for generating such approximations have, especially in recent years, received much attention, both from theoretical and computational viewpoints. See, e.g. [3], [4], [5] and [6].

The analysis of finite element methods for the approximate solution of the stationary Navier-Stokes equations may be viewed as having three components. The first consists of assuming that the finite element subspaces satisfy certain stability and continuity conditions, from which one then deduces estimates for the deviation of the approximate solution from the true solution. The second component then requires us to show that particular finite element subspaces, or classes of subspaces, satisfy the assumed stability and continuity conditions. The final component then requires one to study the computational efficiency of implementations of given finite element methods, in particular as they relate to the solution of the discrete set of nonlinear equations. In this work we are concerned mainly with the first component and somewhat with the third. As will be seen below, the only conditions which need to be verified for a given finite element discretization involve the weak form of the continuity equation. Fortunately, these conditions are identical to those which arise in the context of the linear Stokes equations and, for many different finite element subspaces, have been successfully analyzed. See, e.g. [5], [7], [8], [9], [10], [11] and

[12]. In particular, we point out that [10] and [12] consider elements which, in conjunction with the error estimates derived below, yield optimally accurate velocities and pressures, and which also deal with the very efficient element used to generate the example computations given in this work.

In this work we consider conforming mixed finite element methods for the approximate solution of the inhomogeneous stationary Navier-Stokes equations in bounded regions in $\mathbb{R}^2$ or $\mathbb{R}^3$. These equations are given by (2.1)-(2.3) below. We approximate only when the Navier-Stokes equations possess a unique solution. By inhomogeneous we mean that the momentum equation (2.1) contains a body force, that the boundary condition (2.3) is inhomogeneous and that the continuity equation (2.2) contains a source term. The first of these is included in previous analyses and for internal flows, i.e. flows in bounded regions, is often assumed to vanish or, at most, to be the constant gravitational acceleration. The second of these, namely the inhomogeneous boundary condition, is crucial in internal flows since they are invariably driven by such conditions. This is true both for fictitious, but popular, flows such as the driven cavity, as well as for real internal flows such as those found in ducts which are driven by inflows and outflows. The analysis of such flows is the main concern of this paper. The inhomogeneity in the continuity equation (2.2) requires some comment since, strictly speaking, one cannot have mass sources in incompressible flow. Indeed, the presence of such a source contradicts the very definition of an incompressible flow. We include this inhomogeneity here because, in practice, it is often used in spite of the above inconsistency, e.g. [3] in the context of Stokes flow, because it is often artificially introduced when simplifying boundary conditions, and because it poses no substantial mathematical difficulty.

In typical problems, one may be interested in the flow field itself or, on the other hand, some functional of the velocity or pressure. For example, in ocean circulation problems, one is mainly interested in the flow field itself, and thus the $\underline{L}^2$-norm is a physically interesting norm in which to measure the velocity errors. In other applications, e.g. aerodynamics and duct flows, one may be interested in the pressure or in the derivatives of the velocity since these determine the pressure and viscous forces, respectively, on bodies or walls. In these cases, the $\underline{H}^1$-norm of the velocity error and the L^2-norm of the pressure error are of physical interest. (These and other notations are defined below.) We choose not to use the "eyeball norm" wherein computational results on a fixed grid are plotted and the reader is subsequently asked to agree that the resulting picture is "reasonable". This process can and often is very misleading in the sense that "reasonable" pictures can contain large errors in norms such as the L^2 and H^1-norms. For example, it is relatively easy to generate solutions to the driven cavity problem which display global vortical features that render any picture of the computed flow as being "reasonable" in appearance, while the computed solution itself may be grossly inaccurate in any precise measure of the error.

In the remainder of this section we establish the notation used in the subsequent sections. In Section 2, we present results concerning the exact solution of a particular weak formulation of the stationary Navier-Stokes equations. We give details only when the result is a substantial departure, usually due to the inhomogeneous boundary condition, from known results. In Section 3, we present a finite element algorithm for the approximate solution of our weak problem and present estimates for the error. We also discuss various algorithms for solving the discrete system of algebraic equations which result from the application of the finite element algorithm. Finally, in Section 4, we give some numerical results, mainly with the goal of illustrating the analytical results of Section 3.

1.1 - Notation

Throughout this work Ω will denote a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a Lipschitz continuous boundary Γ. The unit outer normal to Ω will be denoted by $\underline{n}$. $H^r(\Omega)$, for $r \geq 0$ an integer, denotes the Sobolev space of real valued functions with square integrable derivatives of order up to r, equipped with the usual norm. See [13]. We will denote $H^0(\Omega)$ by $L^2(\Omega)$. $\underline{H}^r(\Omega)$ and $\underline{L}^2(\Omega)$ will denote the spaces of vector valued functions each of whose n components, n = 2 or 3, belongs to $H^r(\Omega)$ and $L^2(\Omega)$, respectively. We also define, in the usual manner, the Sobolev spaces $H^r(\Omega)$ for $r < 0$ and the trace spaces $H^s(\Gamma)$ of functions defined on the boundary. Again, see [13] for details. Finally, we define the constrained spaces

$$\underline{H}^1_0(\Omega) \equiv \{\underline{v} \in \underline{H}^1(\Omega): \ \underline{v} = 0 \text{ on } \Gamma\}$$

$$L^2_0(\Omega) \equiv \{\psi \in L^2(\Omega): \int_\Omega \psi \, d\Omega = 0\}.$$

Thus $L^2_0(\Omega)$ consists of $L^2(\Omega)$ functions with zero mean over Ω.

We define the L^2 inner products

$$\langle f,g \rangle \equiv \int_\Omega f g \, d\Omega,$$

$$\langle \underline{u},\underline{v} \rangle \equiv \int_\Omega \underline{u} \cdot \underline{v} \, d\Omega \equiv \int_\Omega u_i v_i \, d\Omega,$$

and

$$\langle \sigma,\tau \rangle \equiv \int_\Omega \sigma : \tau \, d\Omega \equiv \int_\Omega \sigma_{ij} \tau_{ij} \, d\Omega.$$

Here f and g are scalar functions, $\underline{u}$ and $\underline{v}$ are vector valued functions and σ and τ are tensor functions. The colon will denote the scalar product of the two tensors on each side of it. Repeated indices, unless otherwise noted, imply summation over $1,\ldots,n$, where $n = 2$ or 3.

For functions $\underline{v} \in \underline{H}_0^1(\Omega)$, we will use the norm

$$||\underline{v}||_1^2 \equiv \int_\Omega \nabla\underline{v} : \nabla\underline{v}\, d\Omega = \langle \nabla\underline{v}, \nabla\underline{v} \rangle$$

while for functions $\underline{v} \in \underline{H}^1(\Omega)$, we will use the norm

$$||\underline{v}||_1^2 \equiv \langle \nabla\underline{v}, \nabla\underline{v} \rangle + \langle \underline{v}, \underline{v} \rangle .$$

We use the same notation for both norms; which one is actually being used in a particular situation will be clear from the context.

Boundary norms will be denoted by, e.g. $||\cdot||_{1/2,\Gamma}$. Whenever the Γ is omitted, the norm is one for functions defined over Ω.

2. THE INHOMOGENEOUS STATIONARY NAVIER-STOKES EQUATIONS

2.1 - Reduction to a Homogeneous Problem

The Navier-Stokes equations for the velocity $\underline{u}$ and pressure p are given by

$$-\nu\Delta\underline{u} + \underline{u}\cdot\text{grad}\,\underline{u} + \text{grad}\,p = \underline{f} \quad \text{in} \quad \Omega \tag{2.1}$$

$$\text{div}\,\underline{u} = g \quad \text{in} \quad \Omega \tag{2.2}$$

$$\underline{u} = \underline{q} \quad \text{on} \quad \Gamma \tag{2.3}$$

where ν is the constant inverse Reynolds number and $\underline{f} \in \underline{H}^{-1}(\Omega)$, $g \in L_0^2(\Omega)$ and $\underline{q} \in \underline{H}^{1/2}(\Gamma)$ are given functions such that

$$\int_\Omega g \, d\Omega = \int_\Gamma \underline{q} \cdot \underline{n} \, d\Gamma = 0. \tag{2.4}$$

The weak formulation of (2.1)-(2.4) which we consider is to seek $\underline{u} \in \underline{H}^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that (2.3) is satisfied and

$$a_0(\underline{u},\underline{v}) + a_1(\underline{u},\underline{u},\underline{v}) + b(\underline{v},p) = \langle \underline{f},\underline{v} \rangle \quad \forall \;\; \underline{v} \in \underline{H}_0^1(\Omega) \tag{2.5}$$

$$b(\underline{u},\psi) = -\langle g,\psi \rangle \quad \forall \;\; \psi \in L_0^2(\Omega) \tag{2.6}$$

where

$$a_0(\underline{u},\underline{v}) \equiv \nu \int_\Omega \operatorname{grad} \underline{u} : \operatorname{grad} \underline{v} \, d\Omega - \frac{1}{2} \int_\Omega g \underline{u} \cdot \underline{v} \, d\Omega, \tag{2.7}$$

$$a_1(\underline{w},\underline{u},\underline{v}) \equiv \frac{1}{2} \left\{ \int_\Omega \underline{w} \cdot \operatorname{grad} \underline{u} \cdot \underline{v} \, d\Omega - \int_\Omega \underline{w} \cdot \operatorname{grad} \underline{v} \cdot \underline{u} \, d\Omega \right\}, \tag{2.8}$$

and

$$b(\underline{v},\psi) \equiv -\int_\Omega \psi \operatorname{div} \underline{v} \, d\Omega. \tag{2.9}$$

We note that although the test function ψ in (2.6) is in $L_0^2(\Omega)$, (2.6) actually holds for all functions $\psi \in L^2(\Omega)$ because of (2.4).

The motivation for choosing the particular weak formulation (2.5) is twofold. First, we note that if (2.2) holds, then by the divergence theorem we have, for $\underline{u} \in \underline{H}^1(\Omega)$ and $\underline{v} \in \underline{H}_0^1(\Omega)$, that (2.5) is equivalent to

$$\nu\int_\Omega \operatorname{grad} \underline{u} : \operatorname{grad} \underline{v}\, d\Omega + \int_\Omega \underline{u}\cdot\operatorname{grad} \underline{u}\cdot\underline{v}\, d\Omega + b(\underline{v},p) = \langle \underline{f},\underline{v}\rangle \;\; \forall\; \underline{v} \in \underline{H}_0^1(\Omega). \tag{2.10}$$

This is the weak form of (2.1) one would arrive at from a standard Galerkin procedure applied to (2.1). Second, it is obvious from (2.8) that the trilinear form $a_1(\cdot,\cdot,\cdot)$ satisfies the skew-symmetric properties

$$a_1(\underline{w},\underline{u},\underline{v}) = -a_1(\underline{w},\underline{v},\underline{u}) \quad \forall\; \underline{u},\underline{v},\underline{w} \in \underline{H}^1(\Omega) \tag{2.11}$$

$$a_1(\underline{w},\underline{u},\underline{u}) = 0 \qquad \forall\; \underline{u},\underline{w} \in \underline{H}^1(\Omega). \tag{2.12}$$

On the other hand, the trilinear form

$$\tilde{a}_1(\underline{w},\underline{u},\underline{v}) \equiv \int_\Omega \underline{w}\cdot\operatorname{grad} \underline{u}\cdot\underline{v}\, d\Omega \tag{2.13}$$

appearing in (2.10) satisfies (2.11) and (2.12) only when $\underline{u},\underline{v},\underline{w} \in H^1(\Omega)$ with $\underline{w}$ divergence free and at least one of $\underline{u},\underline{v},\underline{w} \in \underline{H}_0^1(\Omega)$. Indeed, for such $\underline{u},\underline{v},\underline{w}$ we have, from (2.8) and (2.13)

$$\tilde{a}_1(\underline{w},\underline{u},\underline{v}) = a_1(\underline{w},\underline{u},\underline{v}). \tag{2.14}$$

The fact that the properties (2.11) and (2.12) hold for the trilinear form (2.8) on all of $\underline{H}^1(\Omega)$ will be useful in the subsequent analyses.

We wish to reduce the problem (2.3), (2.5) and (2.6) into one for which (2.3) and (2.6) become homogeneous. To this end we write

$$\underline{u} = \underline{w} + \underline{z} + \underline{q}_* \tag{2.15}$$

where $\underline{q}_*$ satisfies

$$\underline{q}_* \in \underline{H}^1(\Omega), \quad \underline{q}_*\Big|_\Gamma = \underline{q} \quad \text{and} \quad b(\underline{q}_*,\psi) = 0 \qquad \forall\ \psi \in L_0^2(\Omega), \tag{2.16}$$

and $\underline{w}$ satisfies

$$\underline{w} \in H_0^1(\Omega), \quad b(\underline{w},\psi) = -\langle g,\psi \rangle \quad \forall\ \psi \in L_0^2(\Omega). \tag{2.17}$$

In the sequel, $\underline{q}_*$ will be required to satisfy an additional property, which we consider below.

If we substitute (2.15) into (2.3), (2.5) and (2.6) and use (2.16) and (2.17), we are led to the following problem for $\underline{z}$ and p: seek $\underline{z} \in \underline{H}_0^1(\Omega)$ and $p \in L_0^2(\Omega)$ such that

$$a_0(\underline{z},\underline{v}) + a_1(\underline{z},\underline{z},\underline{v}) + a_1(\underline{w}+\underline{q}_*,\underline{z},\underline{v}) + a_1(\underline{z},\underline{w}+\underline{q}_*,\underline{v})$$

$$+ b(\underline{v},p) = \langle \underline{f},\underline{v} \rangle - a_0(\underline{w}+\underline{q}_*,\underline{v}) - a_1(\underline{w}+\underline{q}_*,\underline{w}+\underline{q}_*,\underline{v}) \quad \forall\ \underline{v} \in \underline{H}_0^1(\Omega), \tag{2.18}$$

$$b(\underline{z},\psi) = 0 \qquad \forall\ \psi \in L_0^2(\Omega). \tag{2.19}$$

We remark that the decomposition (2.15) need not be explicitly used in computations, but rather $\underline{u}$ and p can be obtained directly from discretizations of (2.3), (2.5) and (2.6). In particular, approximations to the functions $\underline{w}$ and $\underline{q}_*$ need not be explicitly computed.

2.2 - Continuity and Stability Properties

We now list the continuity and stability conditions on the forms $a_0(\cdot,\cdot)$,

$a_1(\cdot,\cdot,\cdot)$ and $b(\cdot,\cdot)$ which will be needed to prove the existence and uniqueness of a solution to (2.3), (2.5) and (2.6). We define the subspaces $\underline{Z}$ and $\underline{W}$ by

$$\underline{Z} \equiv \{\underline{z} \in \underline{H}_0^1(\Omega): \quad b(\underline{z},\psi) = 0 \quad \forall\ \psi \in L_0^2(\Omega)\}$$

and

$$\underline{W} \equiv \underline{Z}^{\perp}$$

where the orthogonality is in $\underline{H}_0^1(\Omega)$.

It is easily seen that

$$|b(\underline{v},\psi)| \leq \sqrt{n}\ ||\underline{v}||_1\ ||\psi||_0 \quad \forall\ \underline{v} \in \underline{H}^1(\Omega), \quad \psi \in L_0^2(\Omega). \tag{2.20}$$

Here $n = 2$ or 3 refers to the number of space dimensions. We note that (2.20) implies that $\underline{Z}$ is a closed subspace of $\underline{H}_0^1(\Omega)$ and thus $\underline{H}_0^1(\Omega) = \underline{Z} \oplus \underline{W}$. In addition we need the stability properties

$$\sup_{\substack{||\psi||_0=1 \\ \psi \in L_0^2(\Omega)}} b(\underline{w},\psi) \geq \gamma ||\underline{w}||_1 \quad \forall\ \underline{w} \in \underline{W} \tag{2.21}$$

and

$$\sup_{\substack{||\underline{w}||_1=1 \\ \underline{w} \in \underline{W}}} b(\underline{w},\psi) \geq \hat{\gamma} ||\psi||_0 \quad \forall\ \psi \in L_0^2(\Omega) \tag{2.22}$$

where $\gamma, \hat{\gamma} > 0$. These properties are established in [14].

From (2.8) and (2.13) we have that

$$a_1(\underline{w},\underline{u},\underline{v}) = \frac{1}{2}\{\tilde{a}_1(\underline{w},\underline{u},\underline{v}) - \tilde{a}_1(\underline{w},\underline{v},\underline{u})\}. \tag{2.23}$$

It is known [4] that, with $N > 0$ and $n = 2$ or 3,

$$|\tilde{a}_1(\underline{w},\underline{u},\underline{v})| \leq N||\underline{u}||_1\ ||\underline{v}||_1\ ||\underline{w}||_1 \qquad \forall\ \underline{u},\underline{v},\underline{w} \in \underline{H}^1(\Omega).$$

Therefore, from (2.23), we have that

$$|a_1(\underline{w},\underline{u},\underline{v})| \leq N||\underline{u}||_1\ ||\underline{v}||_1\ ||\underline{w}||_1 \qquad \forall\ \underline{u},\underline{v},\underline{w} \in \underline{H}^1(\Omega) \tag{2.24}$$

as well.

Using Hölder's inequality, we have that

$$\int_\Omega g u_i v_i \, d\Omega \leq |u_i|_{L^4}\ |g|_{L^2}\ |v_i|_{L^4}, \quad \text{no sum over } i. \tag{2.25}$$

Since for $n = 2$ or 3 and Ω bounded, $H^1(\Omega)$ is continuously imbedded in L^4, (2.25) implies that for all $\underline{u},\underline{v} \in \underline{H}^1(\Omega)$ and $g \in L_0^2(\Omega)$,

$$g u_i v_i \in L^1(\Omega)$$

and

$$\left|\int_\Omega g u_i v_i \, d\Omega\right| \leq C||\underline{u}||_1\ ||\underline{v}||_1\ ||g||_0,$$

where there is no sum over i. Therefore, for some $M > 0$,

$$\left|\int_\Omega g \underline{u}\cdot\underline{v} \, d\Omega\right| \leq 2M||\underline{u}||_1\ ||\underline{v}||_1\ ||g||_0 \qquad \forall\ \underline{u},\underline{v} \in \underline{H}^1(\Omega), \quad g \in L_0^2(\Omega). \tag{2.26}$$

It is easily established that

$$\left|\int_\Omega \operatorname{grad} \underline{u} : \operatorname{grad} \underline{v}\, d\Omega\right| \leq \|\underline{u}\|_1 \, \|\underline{v}\|_1 \quad \forall\ \underline{u},\underline{v} \in \underline{H}^1(\Omega)$$

and

$$\int_\Omega \operatorname{grad} \underline{u} : \operatorname{grad} \underline{u}\, d\Omega \geq \|\underline{u}\|_1^2 \quad \forall\ \underline{u} \in \underline{H}_0^1(\Omega).$$

Combining these with (2.7) and (2.26) easily leads to

$$a_0(\underline{u},\underline{u}) \geq (\nu - M\|g\|_0)\|u\|_1^2 \quad \forall\ \underline{u} \in \underline{H}_0^1(\Omega) \tag{2.27}$$

and

$$|a_0(\underline{u},\underline{v})| \leq (\nu + M\|g\|_0)\|\underline{u}\|_1 \, \|\underline{v}\|_1 \quad \forall\ \underline{u},\underline{v} \in \underline{H}^1(\Omega). \tag{2.28}$$

The continuity and stability properties which will be used in the sequel are given by (2.20), (2.21), (2.22), (2.24), (2.27) and (2.28).

2.3 - Existence, Uniqueness and Regularity

We first establish the existence of a solution of the problem (2.3), (2.5) and (2.6). We shall do so by showing the existence of solutions $\underline{g}_*$, $\underline{w}$ of (2.16), (2.17), respectively, and then showing the existence of a solution $(\underline{z},p)$ of the problem (2.18), (2.19). Then, by (2.15), we will have shown the existence of a solution $(\underline{u},p)$ of (2.3), (2.5) and (2.6).

LEMMA 2.1 - *Given* $\underline{g} \in \underline{H}^{1/2}(\Gamma)$ *satisfying* (2.4), *there exists* $\underline{g}_*$ *satisfying* (2.16). *Moreover, for any* $\varepsilon > 0$, *there exists a particular* $\underline{g}_*$ *satisfying* (2.16) *and*

$$|a_1(\underline{z},\underline{q}_*,\underline{z})| \le \varepsilon ||\underline{z}||_1^2 \qquad \forall\ \underline{z} \in \underline{Z}. \tag{2.29}$$

Proof: The existence of a $\underline{q}_*$ satisfying (2.16) and

$$|\tilde{a}_1(\underline{z},\underline{q}_*,\underline{z})| \le \varepsilon ||\underline{z}||_1^2$$

is established in [1] or [3]. Therefore, by (2.14), the inequality (2.29) is also established. ■

LEMMA 2.2 - *Given* $g \in L_0^2(\Omega)$, *there exists a unique* $\underline{w} \in \underline{W}$ *satisfying* (2.17) *and the estimate*

$$||\underline{w}||_1 \le \frac{1}{\gamma} ||g||_0. \tag{2.30}$$

Proof: The bilinear form $b(\cdot,\cdot)$ satisfies the continuity and stability conditions (2.20)-(2.22) on $\underline{W} \times L_0^2(\Omega)$ and $\langle g,\psi \rangle$ is a bounded linear functional on $L_0^2(\Omega)$. Therefore, the result follows from Babuška's generalization of the Lax-Milgram theorem [15]. ■

LEMMA 2.3 - *Given* $\underline{f} \in \underline{H}^{-1}(\Omega)$, $\underline{q}_*$ *satisfying* (2.16), *and* $\underline{w}$ *satisfying* (2.17). *If*

$$\beta \equiv \nu - (M + \frac{N}{\gamma})||g||_0 > 0, \tag{2.31}$$

then there exists $\underline{z} \in \underline{Z}$ *and* $p \in L_0^2(\Omega)$ *satisfying* (2.18) *and* (2.19) *and the estimates*

$$||\underline{z}||_1 \le \frac{1}{\beta}\left[||\underline{f}||_{-1} + (\nu + M||g||_0)||\underline{w}+\underline{q}_*||_1 + N||\underline{w}+\underline{q}_*||_1^2\right] \tag{2.32}$$

and

$$||p||_0 \leq \frac{1}{\hat{\gamma}}\left[||\underline{f}||_{-1} + (\nu + M||g||_0)||\underline{z}+\underline{w}+\underline{q}_*||_1 + N||\underline{z}+\underline{w}+\underline{q}_*||_1^2\right]. \tag{2.33}$$

Proof: We set $\underline{v} = \underline{\zeta} \in \underline{Z}$ in (2.18). Then, since $b(\underline{\zeta},p) = 0$ for $\underline{\zeta} \in \underline{Z}$, we have

$$\hat{a}(\underline{z},\underline{z},\underline{\zeta}) = h(\underline{\zeta}) \quad \forall\ \underline{\zeta} \in \underline{Z} \tag{2.34}$$

where

$$\hat{a}(\underline{z},\underline{z},\underline{\zeta}) \equiv a_0(\underline{z},\underline{\zeta}) + a_1(\underline{z},\underline{z},\underline{\zeta}) + a_1(\underline{w}+\underline{q}_*,\underline{z},\underline{\zeta})$$
$$+ a_1(\underline{z},\underline{w}+\underline{q}_*,\underline{\zeta}) \tag{2.35}$$

and

$$h(\underline{\zeta}) = \langle \underline{f},\underline{\zeta} \rangle - a_0(\underline{w}+\underline{q}_*,\underline{\zeta}) - a_1(\underline{w}+\underline{q}_*,\underline{w}+\underline{q}_*,\underline{\zeta}). \tag{2.36}$$

Now, from (2.19), we see that $\underline{z} \in \underline{Z}$. To show the existence of a solution $\underline{z}$ to (2.34) we need only show that (see [3] or [4]) there exists $\beta > 0$ such that

$$\hat{a}(\underline{z},\underline{z},\underline{z}) \geq \beta||\underline{z}||_1^2 \quad \forall\ \underline{z} \in \underline{Z} \tag{2.37}$$

and that if $\underline{z}_m$ converges to $\underline{z}$ weakly in $\underline{Z}$ as $m \to \infty$, then

$$\lim_{m\to\infty} \hat{a}(\underline{z}_m,\underline{z}_m,\underline{\zeta}) = \hat{a}(\underline{z},\underline{z},\underline{\zeta}) \quad \forall\ \underline{\zeta} \in \underline{Z} \tag{2.38}$$

and finally that $h(\underline{\zeta})$ is a bounded linear functional on $\underline{Z}$. Now, letting $\underline{\zeta} = \underline{z}$ in (2.35) yields, using (2.12),

$$\hat{a}(\underline{z},\underline{z},\underline{z}) = a_0(\underline{z},\underline{z}) + a_1(\underline{z},\underline{w}+\underline{q}_*,\underline{z}).$$

Then, using (2.24), (2.28) and (2.29),

$$\hat{a}(\underline{z},\underline{z},\underline{z}) \geq (\nu - M||g||_0 - N||\underline{w}||_1 - \varepsilon)||\underline{z}||_1^2, \tag{2.39}$$

or, using (2.30),

$$\hat{a}(\underline{z},\underline{z},\underline{z}) \geq [\nu - (M + \frac{N}{\gamma})||g||_0 - \varepsilon]||\underline{z}||_1^2.$$

Therefore, since $\varepsilon > 0$ is arbitrary, (2.37) holds with β given by (2.31). Note that $\beta > 0$ requires that ν be "sufficiently large" or g "sufficiently small". To prove (2.38), we note that the term $a_1(\underline{z},\underline{z},\underline{\zeta})$ equals $\tilde{a}_1(\underline{z},\underline{z},\underline{\zeta})$ since $\underline{z},\underline{\zeta} \in \underline{Z} \subset \underline{H}_0^1(\Omega)$. The convergence of $\tilde{a}_1(\underline{z}_m,\underline{z}_m,\underline{\zeta})$ to $\tilde{a}_1(\underline{z},\underline{z},\underline{\zeta})$ was established in [3] or [4]. The remaining terms in (2.34), i.e.

$$a_0(\underline{z},\underline{\zeta}) + a_1(\underline{w}+\underline{q}_*,\underline{z},\underline{\zeta}) + a_1(\underline{z},\underline{w}+\underline{q}_*,\underline{\zeta})$$

constitute, by (2.24), (2.28) and (2.30) a continuous bilinear form on $\underline{Z} \times \underline{Z}$. Thus, (2.38) follows. Finally, we have from (2.24), (2.28) and (2.36),

$$h(\underline{\zeta}) \leq \left[||\underline{f}||_{-1} + (\nu + M||g||_0)||\underline{w}+\underline{q}_*||_1 + N||\underline{w}+\underline{q}_*||_1^2\right]||\underline{\zeta}||_1 \quad \forall\ \underline{\zeta} \in \underline{Z},$$

so that $h(\underline{\zeta})$ is a bounded linear functional on $\underline{Z}$. Thus the problem (2.34) has a solution $\underline{z} \varepsilon \underline{Z}$ satisfying (2.32).

Now, set $\underline{v} = \underline{w} \varepsilon \underline{W}$ in (2.18). We then have, by (2.15),

$$b(\underline{w},p) = \langle \underline{f},\underline{w} \rangle - a_0(\underline{u},\underline{w}) - a_1(\underline{u},\underline{u},\underline{w}) \quad \forall\ \underline{w} \ \varepsilon\ \underline{W}. \tag{2.40}$$

Since $||\underline{u}||_1 \leq ||\underline{w}||_1 + ||\underline{z}||_1 + ||\underline{g}_*||_1$ and $\underline{f} \ \varepsilon\ \underline{H}^{-1}(\Omega)$, the right hand side is, by (2.24), (2.28), Lemmata 2.1 and 2.2, and (2.32), a bounded linear functional on $\underline{W}$. Then, since the bilinear form $b(\cdot,\cdot)$ satisfies (2.20)-(2.22), we have that a p exists satisfying (2.40) and the estimate (2.33). ■

THEOREM 2.4 - *Given* $\underline{f} \ \varepsilon\ \underline{H}^{-1}(\Omega)$, $g \ \varepsilon\ L_0^2(\Omega)$ *and* $\underline{q} \ \varepsilon\ \underline{H}^{1/2}(\Gamma)$ *such that (2.4) and (2.31) are satisfied, there exists a solution* $\underline{u} \ \varepsilon\ \underline{H}^1(\Omega)$ *and* $p \ \varepsilon\ L_0^2(\Omega)$ *of (2.3), (2.5) and (2.6). Moreover,* $\underline{u}$ *and* p *satisfy the estimates*

$$||\underline{u}||_1 \leq \frac{1}{\gamma}||g||_0 + ||\underline{g}_*||_1 + \frac{1}{\beta}\Big[||\underline{f}||_{-1} + (\nu + M||g||_0)(\frac{1}{\gamma}||g||_0 + ||q_*||_1)$$

$$+ N(\frac{1}{\gamma}||g||_0 + ||\underline{g}_*||_1)^2\Big] \tag{2.41}$$

and

$$||p||_0 \leq \frac{1}{\hat{\gamma}}\Big[||\underline{f}||_{-1} + (\nu + M||g||_0)||\underline{u}||_1 + N||\underline{u}||_1^2\Big]. \tag{2.42}$$

Proof: The results follow from Lemmata 2.1-2.3. ■

We remark that if $g = 0$, then $\beta = \nu$ and $\beta > 0$ trivially. For $g \neq 0$, (2.31) requires that ν be sufficiently large compared to $||g||_0$.

The uniqueness of the solution $\underline{u} \in \underline{H}^1(\Omega)$ and $p \in L_0^2(\Omega)$ will be shown directly from (2.3), (2.5) and (2.6).

THEOREM 2.5 - *Given* $\underline{f} \in \underline{H}^{-1}(\Omega)$, $g \in L_0^2(\Omega)$ *and* $\underline{q} \in \underline{H}^{1/2}(\Gamma)$ *such that* (2.4) *and*

$$\xi \equiv \nu - (M + \frac{N}{\gamma})||g||_0 - N||\underline{q}_*||_1 - \frac{N}{\beta}\Big\{||\underline{f}||_{-1} + (\nu + M||g||_0)(\frac{1}{\gamma}||g||_0 + ||\underline{q}_*||_1)$$
$$+ N(\frac{1}{\gamma}||g||_0 + ||\underline{q}_*||_1)^2\Big\} > 0 \qquad (2.43)$$

are satisfied, there exists at most one solution $\underline{u} \in \underline{H}^1(\Omega)$ *and* $p \in L_0^2(\Omega)$ *of* (2.3), (2.5) *and* (2.6).

<u>Proof</u>: Let $(\underline{u}_1, p_1)$ and $(\underline{u}_2, p_2)$ be two solutions in $\underline{H}^1(\Omega) \times L_0^2(\Omega)$ of (2.3), (2.5) and (2.6). If $\underline{U} = \underline{u}_1 - \underline{u}_2$ and $P = p_1 - p_2$, we have, from (2.3), (2.5) and (2.6), that $\underline{U} \in \underline{H}_0^1(\Omega)$ and

$$a_0(\underline{U},\underline{v}) + a_1(\underline{u}_1,\underline{u}_1,\underline{v}) - a_1(\underline{u}_2,\underline{u}_2,\underline{v}) + b(\underline{v},\underline{P}) = 0 \qquad \forall\ \underline{v} \in \underline{H}_0^1(\Omega) \qquad (2.44)$$

$$b(\underline{U},\psi) = 0 \qquad \forall\ \psi \in L_0^2(\Omega). \qquad (2.45)$$

Choosing $\psi = \underline{P}$ in (2.45) and $\underline{v} = \underline{U} \in \underline{H}_0^1(\Omega)$ in (2.44) yields

$$a_0(\underline{U},\underline{U}) + a_1(\underline{U},\underline{u}_1,\underline{U}) + a_1(\underline{u}_2,\underline{U},\underline{U}) = 0.$$

Then, by (2.12), we have

$$a_0(\underline{U},\underline{U}) + a_1(\underline{U},\underline{u}_1,\underline{U}) = 0$$

so that by (2.24) and (2.27),

$$(\nu - M||g||_0 - N||\underline{u}||_1)||\underline{U}||_1^2 \leq 0.$$

Thus, using (2.41), if (2.43) holds, $\underline{U} = 0$.

Now, with $\underline{U} = 0$ and $\underline{u}_1 = \underline{u}_2$, (2.44) becomes

$$b(\underline{v},\underline{P}) = 0 \quad \forall\ \underline{v} \in \underline{H}_0^1(\Omega)$$

and, in particular, since $\underline{W} \subset \underline{H}_0^1(\Omega)$,

$$b(\underline{w},\underline{P}) = 0 \quad \forall\ \underline{w} \in \underline{W}.$$

Then (2.22) implies that $P = 0$. ■

We note that if $g = 0$ and $\underline{q} = 0$, then (2.43) reduces to the well known uniqueness condition $\nu^2 - N||\underline{f}||_{-1} > 0$. Moreover, for existence we needed $\beta > 0$, while for uniqueness, $\xi > 0$ was required. Comparing (2.31) and (2.43), we see that $\beta > \xi$, so that $\xi > 0$ implies $\beta > 0$, but not conversely.

The condition for uniqueness, (2.43), requires that $\underline{f}$, g, and $\underline{q}$ be "sufficiently small". On the other hand, the condition for existence, (2.31), requires only that g be "sufficiently small". If one is willing to accept "small" $\underline{q}$, as one must to prove uniqueness, then the existence proof can be simplified. Moreover, it can be modified so that $\underline{q}_*$ need not be divergence free and therefore g need not have zero mean. Furthermore, $\underline{q}_*$ need not satisfy (2.29). Indeed, we replace (2.4), (2.6) and (2.17) by

$$\int_\Omega g\, d\Omega = \int_\Gamma \underline{q}\cdot\underline{n}\, d\Gamma, \tag{2.46}$$

$$\underline{q}_* \in \underline{H}^1(\Omega), \quad \underline{q}_*\big|_\Gamma = \underline{q}, \tag{2.47}$$

and

$$\underline{w} \in \underline{H}_0^1(\Omega), \quad b(\underline{w},\psi) = \langle \operatorname{div} \underline{q}_* - g, \psi \rangle \quad \forall\, \psi \in L_0^2(\Omega), \tag{2.48}$$

respectively. We replace Lemma 2.1 by the fact that, since $\underline{q} \in \underline{H}^{1/2}(\Gamma)$, there exists a $\underline{q}_* \in \underline{H}^1(\Omega)$ such that $\underline{q}_*\big|_\Gamma = \underline{q}$ and

$$||\underline{q}_*||_1 \leq K||\underline{q}||_{1/2,\Gamma}. \tag{2.49}$$

Through the use of (2.24) and (2.49), we replace (2.29) by

$$a_1(\underline{z},\underline{q}_*,\underline{z}) \leq N||\underline{q}_*||_1\ ||\underline{z}||_1^2 \leq NK||\underline{z}||_1^2\ ||\underline{q}||_{1/2,\Gamma}. \tag{2.50}$$

Lemmata 2.2 and 2.3 are replaced by analogous results, which are proven in much the same way as the lemmata they replace. We again note that whenever (2.29) was needed above, it is replaced by (2.50). Then theorems analogous to Theorems 2.4 and 2.5 follow. Here we simply list the results which replace Lemma 2.2 and Theorems 2.4 and 2.5.

LEMMA 2.6 - *Given* $g \in L^2(\Omega)$ *and* $\underline{q}_* \in H^1(\Omega)$ *satisfying* (2.46) *and* (2.47), *there exists a unique* $\underline{w} \in \underline{W}$ *satisfying* (2.48) *and the estimate*

$$||\underline{w}||_1 \leq \frac{1}{\gamma}(||g||_0 + ||\mathrm{div}\, \underline{g}_*||_0) \leq \frac{1}{\gamma}(||g||_0 + \sqrt{n}\, ||\underline{g}_*||_1). \qquad (2.51)$$

THEOREM 2.7 - *Given* $\underline{f} \in \underline{H}^{-1}(\Omega)$, $g \in L^2(\Omega)$ *and* $\underline{g} \in H^{1/2}(\Gamma)$ *such that* (2.46) *and*

$$\hat{\beta} \equiv \nu - (M + \frac{N}{\gamma})||g||_0 - NK(1 + \frac{\sqrt{n}}{\gamma})||\underline{g}||_{1/2,\Gamma} > 0 \qquad (2.52)$$

are satisfied, there exists a solution $\underline{u} \in \underline{H}^1(\Omega)$ *and* $p \in L_0^2(\Omega)$ *of* (2.3), (2.5) *and* (2.6). *Moreover,* $\underline{u}$ *satisfies the estimate*

$$||\underline{u}||_1 \leq \frac{1}{\gamma}||g||_0 + (1 + \frac{\sqrt{n}}{\gamma})K||\underline{g}||_{1/2,\Gamma} + B/\hat{\beta} \qquad (2.53)$$

where

$$B \equiv ||\underline{f}||_{-1} + (\nu + M||g||_0)\left\{\frac{1}{\gamma}||g||_0 + (1 + \frac{\sqrt{n}}{\gamma})K||\underline{g}||_{1/2,\Gamma}\right\}$$
$$+ N\left\{\frac{1}{\gamma}||g||_0 + (1 + \frac{\sqrt{n}}{\gamma})K||\underline{g}||_{1/2,\Gamma}\right\}^2 \qquad (2.54)$$

and p *satisfies the estimate* (2.42).

THEOREM 2.8 - *Given* $\underline{f} \in \underline{H}^{-1}(\Omega)$, $g \in L^2(\Omega)$ *and* $\underline{g} \in \underline{H}^{1/2}(\Gamma)$ *such that* (2.46) *and*

$$\hat{\xi} \equiv \hat{\beta} - \frac{NB}{\hat{\beta}} > 0 \qquad (2.55)$$

are satisfied, there exists at most one solution $\underline{u} \in \underline{H}^1(\Omega)$ *and* $p \in L_0^2(\Omega)$ *of (2.3), (2.5) and (2.6).*

We remark that if $\underline{q}_*$ is chosen to be divergence free, which is always possible if g has zero mean over Ω, then all the terms in (2.51)-(2.55) involving

$$\sqrt{n}\ ||\underline{q}||_{1/2,\Gamma}$$

can be omitted. Further, it is clear from (2.52) that existence is proven for "sufficiently small" g and $\underline{q}$ and from (2.55) that uniqueness is proven for "sufficiently small" $\underline{f}$, g and $\underline{q}$. It is also easily shown that uniqueness implies existence, i.e. that $\hat{\xi} > 0$ implies $\hat{\beta} > 0$.

The regularity results for the solution of the stationary Navier-Stokes equations proven in, e.g. [1] or [3], are independent of the particular weak form of the equations, and thus carry over to our setting.

3. MIXED FINITE ELEMENT APPROXIMATIONS

3.1 - The Approximate Problem

We wish to define a problem which will yield approximate solutions of (2.3)-(2.6). To this end we choose subspaces $\underline{V}^h \subset \underline{H}^1(\Omega)$ and $S_0^h \subset L_0^2(\Omega)$. We then seek a $\underline{u}^h \in \underline{V}^h$ and $p^h \in S_0^h$ such that

$$a_0(\underline{u}^h,\underline{v}^h) + a_1(\underline{u}^h,\underline{u}^h,\underline{v}^h) + b(\underline{v}^h,p^h) = \langle \underline{f},\underline{v}^h \rangle \quad \forall\, \underline{v}^h \in \underline{V}_0^h \tag{3.1}$$

$$b(\underline{u}^h,\psi^h) = -\langle g,\psi^h \rangle \quad \forall\, \psi^h \in S_0^h \tag{3.2}$$

$$\underline{u}^h = \underline{q}^h \qquad \text{on } \Gamma \tag{3.3}$$

where $\underline{V}_0^h = \underline{V}^h \cap \underline{H}_0^1(\Omega)$ and $\underline{q}^h$ is an approximation to $\underline{q}$ on Γ. Since we are assuming that $\underline{q}^h$ is in the restriction of $\underline{V}^h$ to the boundary Γ, the results below will hold for polygonal domains. However, through the use of isoparametric elements, it is reasonable to expect that these results can be extended to regions with curved boundaries. Possible choices for $\underline{q}^h$ are the interpolant of $\underline{q}$ in the restriction of $\underline{V}^h$ to Γ or the $\underline{L}^2(\Gamma)$-projection of $\underline{q}$ into that boundary space. The first choice requires that $\underline{q} \in H^{1/2+\varepsilon}(\Gamma)$ for some $\varepsilon > 0$, while for the latter choice $\underline{q} \in H^{1/2}(\Gamma)$ suffices.

In analogy with (2.15), we write

$$\underline{u}^h = \underline{w}^h + \underline{z}^h + \underline{q}_*^h \tag{3.4}$$

where $\underline{q}_*^h \in \underline{V}^h$ is a function such that $\underline{q}_*^h = \underline{q}^h$ on Γ and where $\underline{w}^h$ satisfies

$$\underline{w}^h \in \underline{V}_0^h, \quad b(\underline{w}^h,\psi^h) = \langle \operatorname{div} \underline{q}_*^h - g,\psi^h \rangle \quad \forall\, \psi^h \in S_0^h. \tag{3.5}$$

We only consider the case where the continuous problem has a unique solution. Thus, as in that case, we do not need to require

$$b(\underline{q}_*^h,\psi^h) = 0 \quad \forall\, \psi^h \in S_0^h$$

in order to prove the existence of a solution to (3.1)-(3.3). Furthermore, we will not need to require that $\underline{g}_*^h$ satisfy a result analogous to Lemma 2.1.

The coercivity and continuity conditions (2.20), (2.24), (2.27) and (2.28) hold on the subspaces. However, the conditions (2.21) and (2.22) do not imply that similar conditions hold on the subspaces. Therefore, defining

$$\underline{Z}^h \equiv \{\underline{z}^h \in \underline{V}_0^h: \quad b(\underline{z}^h,\psi^h) = 0 \qquad \forall\, \psi^h \in S_0^h\}$$

and

$$\underline{W}^h \equiv (\underline{Z}^h)^\perp$$

we assume that, for $\gamma_h, \hat{\gamma}_h > 0$,

$$\sup_{\substack{||\psi^h||_0=1 \\ \psi^h \in S_0^h}} b(\underline{w}^h,\psi^h) \geq \gamma_h||\underline{w}^h||_1 \qquad \forall\, \underline{w}^h \in \underline{W}^h \tag{3.6}$$

and

$$\sup_{\substack{||\underline{w}^h||_1=1 \\ \underline{w}^h \in \underline{W}^h}} b(\underline{w}^h,\psi^h) \geq \hat{\gamma}_h||\psi^h||_0 \qquad \forall\, \psi^h \in S_0^h. \tag{3.7}$$

A variety of finite element spaces for which (3.6) and (3.7) hold, with γ_h and $\hat{\gamma}_h$ bounded below uniformly in h, have been analyzed in, e.g. [4], [9], [10], [11] and [12].

We note that, in general, $\underline{Z}^h \not\subset \underline{Z}$. A measure of the angle between the spaces $\underline{Z}^h$ and $\underline{Z}$ is given by

$$\Theta \equiv \sup_{\substack{\underline{z}^h \in \underline{Z}^h \\ ||\underline{z}^h||_1 = 1}} \inf_{\underline{z} \in \underline{Z}} ||\underline{z}-\underline{z}^h||_1. \tag{3.8}$$

In general, $0 \leq \Theta \leq 1$, which is easily seen by observing that for $\underline{Z}^h \subset \underline{Z}$, $\Theta = 0$, and that by choosing $\underline{z} = 0$, $\Theta = 1$.

We are now in a position to prove uniqueness and existence theorems for the approximate solution $\underline{u}^h$, p^h analogous to Theorems 2.7 and 2.8 for the continuous problem.

THEOREM 3.1 - *Given* $\underline{f} \in H^{-1}(\Omega)$, $g \in L^2(\Omega)$ *and* $\underline{g}_*^h \in \underline{V}^h$ *such that* $\underline{g}_*^h = \underline{g}^h$ *on* Γ *and*

$$\hat{\beta}_h = \nu - (M + \frac{N}{\gamma_h})||g||_0 - N(1 + \frac{\sqrt{n}}{\gamma_h})||\underline{g}_*^h||_1 > 0 \tag{3.9}$$

are satisfied, there exists a solution $\underline{u}^h \in \underline{V}^h$ *and* $p^h \in S_0^h$ *of (3.1)-(3.3). Moreover,* $\underline{u}^h$ *and* p^h *satisfy the estimates*

$$||\underline{u}^h||_1 \leq \frac{1}{\gamma_h}||g||_0 + (1 + \frac{\sqrt{n}}{\gamma_h})||\underline{g}_*^h||_1 + B_h/\hat{\beta}_h \tag{3.10}$$

and

$$||p^h||_0 \leq \frac{1}{\hat{\gamma}_h}\left[||\underline{f}||_{-1} + (\nu+M||g||_0)||\underline{u}^h||_1 + N||\underline{u}^h||_1^2\right] \tag{3.11}$$

where

$$B_h = ||\underline{f}||_{-1} + (\nu+M||g||_0)\left[\frac{1}{\gamma_h}||g||_0 + (1 + \frac{\sqrt{n}}{\gamma_h})||\underline{g}_*^h||_1\right]$$
$$+ N\left[\frac{1}{\gamma_h}||g||_0 + (1 + \frac{\sqrt{n}}{\gamma_h})||\underline{g}_*^h||_1\right]^2. \tag{3.12}$$

THEOREM 3.2 - *Given* $\underline{f} \in \underline{H}^{-1}(\Omega)$, $g \in L^2(\Omega)$ *and* $\underline{q}_*^h \in \underline{V}^h$ *such that* $\underline{q}_*^h = \underline{q}^h$ *on* Γ *and*

$$\hat{\xi}_h \equiv \hat{\beta}_h - \frac{NB_h}{\hat{\beta}_h} > 0 \tag{3.13}$$

are satisfied, there exists at most one solution $\underline{u}^h \in \underline{V}^h$ *and* $p^h \in S_o^h$ *of (3.1)-(3.3).*

The proofs of these theorems proceed as in the continuous case with the exception that in the proof of the result analogous to Lemma 2.3, we need not pass to the limit $m \to \infty$, i.e. we need not prove (2.38).

If $\underline{q}_*^h$ is chosen to be "discretely divergence free", i.e. if

$$b(\underline{q}_*^h, \psi^h) = 0 \qquad \forall\ \psi^h \in S_o^h, \tag{3.14}$$

then all terms involving $\sqrt{n}$ in (3.9), (3.10) and (3.12) may be omitted. On the other hand, if $\underline{q}_*^h$ is chosen to be the interpolant, in $\underline{V}^h$, of $\underline{q}_*$, and $\underline{q}_*$ is chosen to satisfy (2.49), then for certain classes of finite element spaces

$$||\underline{q}_*^h||_1 \leq ||\underline{q}_* - \underline{q}_*^h||_1 + ||\underline{q}_*||_1 \leq (1+C)K||\underline{q}||_{1/2,\Gamma}$$

so that if $\underline{q}$ is "small", so will be $\underline{q}_*^h$. Furthermore, if $\underline{q}$ is smoother on the boundary, as it will have to be in order to obtain any degree of approximation, we have that the above inequality holds for a general finite element space $\underline{V}^h$, so that again, $\underline{q}$ "small" implies "$\underline{q}_*^h$" small.

In practice, we compute directly with (3.1)-(3.3) so that we do not need to explicitly construct $\underline{g}_*$ or $\underline{g}_*^h$. However, we do need $\underline{g}^h$, which is defined on Γ.

3.2 - The $\underline{H}^1$ Velocity and L^2 Pressure Error Estimates

We define the set $\hat{\underline{V}}^h$ by

$$\hat{\underline{V}}^h \equiv \{\hat{\underline{u}}^h \in \underline{V}^h : \hat{\underline{u}}^h = \underline{g}^h \text{ on } \Gamma\}.$$

The main goal of this paper is the following result.

THEOREM 3.3 - *Let (2.55) and (3.13) be satisfied so that* $(\underline{u},p)$ *and* $(\underline{u}^h,p^h)$ *denote the unique solutions of (2.3), (2.5) and (2.6) and (3.1)-(3.3), respectively. Then, there exist constants* C_i, $i = 1,\ldots,4$ *such that*

$$||\underline{u}-\underline{u}^h||_1 \leq C_1 \inf_{\hat{\underline{u}}^h \in \hat{\underline{V}}^h} ||\underline{u}-\hat{\underline{u}}^h||_1 + C_2\Theta \inf_{\hat{p}^h \in S_0^h} ||p-\hat{p}^h||_0 \tag{3.15}$$

and

$$||p-p^h||_0 \leq C_3 \inf_{\hat{u}^h \in \hat{\underline{V}}^h} ||\underline{u}-\hat{\underline{u}}^h||_1 + C_3 \inf_{\hat{p}^h \in S_0^h} ||p-\hat{p}^h||_0, \tag{3.16}$$

where Θ *is defined by (3.8).*

Proof: Let $\underline{u}$ and $\underline{u}^h$ be written as in (2.15) and (3.4), respectively, where $\underline{w} \in \underline{W}$, $\underline{w}^h \in \underline{W}^h$, $\underline{z} \in \underline{Z}$, $\underline{z}^h \in Z^h$, $\underline{g}_*\big|_\Gamma = \underline{g}$, and $\underline{g}_*^h\big|_\Gamma = \underline{g}^h$. Then, subtracting (3.2) from (2.6) with $\psi = \psi^h \in S_0^h$ yields

$$b(\underline{w}-\underline{w}^h,\psi^h) = -b(\underline{q}_*-\underline{q}_*^h,\psi^h) \quad \forall\ \psi^h \in S_o^h, \tag{3.17}$$

where we have used $b(\underline{z}-\underline{z}^h,\psi^h) = 0$. Now, let $\hat{\underline{u}}^h = \hat{\underline{w}}^h + \hat{\underline{z}}^h + \underline{q}_*^h$ where $\hat{\underline{w}}^h \in \underline{W}^h$ and $\hat{\underline{z}}^h \in \underline{Z}^h$ are arbitrary. Then $\hat{\underline{u}}^h \in \hat{\underline{V}}^h$ and, since $b(z-\hat{z}^h,\psi^h) = 0$, we have from (3.17)

$$b(\hat{\underline{w}}^h-\underline{w}^h,\psi^h) = b(\hat{\underline{w}}^h+\underline{q}_*^h-\underline{w}-\underline{q}_*,\psi^h) = b(\hat{\underline{u}}^h-\underline{u},\psi^h) \quad \forall\ \psi^h \in S_o^h.$$

Then, using (2.20) and (3.6),

$$||\underline{w}^h-\hat{\underline{w}}^h||_1 \leq \frac{\sqrt{n}}{\gamma_h}\, ||\underline{u}-\hat{\underline{u}}^h||_1 \tag{3.18}$$

where $\hat{\underline{u}}^h$ is an arbitrary element of $\hat{\underline{V}}^h$.

We now estimate $||\underline{z}^h-\hat{\underline{z}}^h||_1$. Subtracting (3.1) from (2.5) with $\underline{v} = \underline{v}^h \in \underline{V}_o^h$ yields

$$a_o(\underline{u}-\underline{u}^h,v^h) + a_1(\underline{u}^h,\underline{u}-\underline{u}^h,v^h) + a_1(\underline{u}-\underline{u}^h,\underline{u},\underline{v}^h) + b(\underline{v}^h,p-p^h) = 0 \quad \forall\ \underline{v}^h \in \underline{V}_o^h \tag{3.19}$$

or, with $\hat{\underline{u}}^h$ arbitrary in $\hat{\underline{V}}^h$ and $\hat{p}^h$ arbitrary in S_o^h,

$$\begin{aligned} &a_o(\hat{\underline{u}}^h-\underline{u}^h,\underline{v}^h) + a_1(\underline{u}^h,\hat{\underline{u}}^h-\underline{u}^h,\underline{v}^h) + a_1(\hat{\underline{u}}^h-\underline{u}^h,\underline{u},\underline{v}^h) + b(\underline{v}^h,\hat{p}^h-p^h) \\ &\quad = a_o(\hat{\underline{u}}^h-\underline{u},\underline{v}^h) + a_1(\underline{u}^h,\hat{\underline{u}}^h-\underline{u},\underline{v}^h) + a_1(\hat{\underline{u}}^h-\underline{u},\underline{u},\underline{v}^h) + b(v^h,\hat{p}^h-p) \\ &\qquad\qquad \forall\ \underline{v}^h \in \underline{V}_o^h. \end{aligned} \tag{3.20}$$

Now choose $\underline{v}^h = \hat{\underline{z}}^h-\underline{z}^h \in \underline{Z}^h \subset \underline{V}_o^h$. Then, with $\underline{\zeta}$ arbitrary in $\underline{Z}$,

$b(\hat{\underline{z}}^h-\underline{z}^h,\hat{p}^h-p^h) = 0$ and

$$b(\hat{\underline{z}}^h-\underline{z}^h,\hat{p}^h-p) = b(\hat{\underline{z}}^h-\underline{z}^h-\underline{\zeta},\hat{p}^h-p).$$

Combining this result with (3.20) produces

$$a_0(\hat{\underline{z}}^h-\underline{z}^h,\hat{\underline{z}}^h-\underline{z}^h) + a_1(\hat{\underline{z}}^h-\underline{z}^h,\underline{u},\hat{\underline{z}}^h-\underline{z}^h) + a_1(\underline{u}^h,\hat{\underline{z}}^h-\underline{z}^h,\hat{\underline{z}}^h-\underline{z}^h)$$

$$= a_0(\hat{\underline{u}}^h-\underline{u}+\underline{w}^h-\hat{\underline{w}}^h,\hat{\underline{z}}^h-\underline{z}^h) + a_1(\hat{\underline{u}}^h-\underline{u}+\underline{w}^h-\hat{\underline{w}}^h,\underline{u},\hat{\underline{z}}^h-\underline{z}^h)$$

$$+ a_1(\underline{u}^h,\hat{\underline{u}}^h-\underline{u}+\underline{w}^h-\hat{\underline{w}}^h,\hat{\underline{z}}^h-\underline{z}^h) + b(\hat{\underline{z}}^h-\underline{z}^h-\underline{\zeta},\hat{p}^h-p)$$

since $\hat{\underline{u}}^h - \underline{u}^h = \hat{\underline{w}}^h - \underline{w}^h + \hat{\underline{z}}^h - \underline{z}^h$. Using (2.12), (2.20), (2.24), (2.27) and (2.28) we obtain

$$(\nu-M||g||_0-N||\underline{u}||_1)||\hat{\underline{z}}^h-\underline{z}^h||_1^2 \leq \sqrt{n}\ ||\hat{p}^h-p||_0\ ||\hat{\underline{z}}^h-\underline{z}^h-\underline{\zeta}||_1$$

$$+\Big((\nu+M||g||_0+N||\underline{u}||_1+N||\underline{u}^h||_1)(||\hat{\underline{u}}^h-\underline{u}||_1+||\hat{\underline{w}}^h-\underline{w}^h||_1)\Big)||\hat{\underline{z}}^h-\underline{z}^h||_1.$$

Since $\hat{\xi} > 0$ from (2.55), we have, by taking the infimum over $\underline{\zeta} \in \underline{Z}$ and using (3.18),

$$||\hat{\underline{z}}^h-\underline{z}^h||_1 \leq \frac{1}{\hat{\xi}}\left\{\Big((\nu+M||g||_0+N||\underline{u}||_1+N||\underline{u}^h||_1)(1 + \frac{\sqrt{n}}{\gamma_h}\Big)||\underline{u}-\hat{\underline{u}}^h||_1\right.$$

$$\left. + \sqrt{n}\ ||p-\hat{p}^h||_0 \inf_{\underline{\zeta}\in\underline{Z}}\Big[||\hat{\underline{z}}^h-\underline{z}^h-\underline{\zeta}||_1/||\hat{\underline{z}}^h-\underline{z}^h||_1\Big]\right\}.$$

Then, since

$$\inf_{\underline{\zeta}\epsilon\underline{Z}} \left\{ ||\hat{\underline{z}}^h - \underline{z}^h - \underline{\zeta}||_1 / ||\hat{\underline{z}}^h - \underline{z}^h||_1 \right\} \leq \Theta,$$

we have that

$$||\hat{\underline{z}}^h - \underline{z}^h||_1 \leq \frac{1}{\hat{\xi}} \left\{ \left((\nu + M||g||_o + N||\underline{u}||_1 + N||\underline{u}^h||_1)(1 + \frac{\sqrt{n}}{\gamma_h} \right) ||\underline{u} - \hat{\underline{u}}^h||_1 \right.$$

$$\left. + \sqrt{n}\ \Theta\ ||p - \hat{p}^h||_o \right\} . \tag{3.21}$$

Now, since

$$||\underline{u} - \underline{u}^h||_1 \leq ||\underline{u} - \hat{\underline{u}}^h||_1 + ||\hat{\underline{u}}^h - \underline{u}^h||_1 \leq ||\underline{u} - \hat{\underline{u}}^h||_1 + ||\hat{\underline{w}}^h - \underline{w}^h||_1 + ||\hat{\underline{z}}^h - \underline{z}^h||_1$$

we have, using (3.18) and (3.21) and taking the infimum over $\hat{p}^h \in S_o^h$ and $\hat{\underline{u}}^h \in \hat{\underline{V}}^h$ that (3.15) holds. Moreover, using (2.53) and (3.10) yields that

$$C_1 = 1 + \frac{\sqrt{n}}{\gamma_h} + \frac{1}{\hat{\xi}} \left((\nu + M||g||_o + \frac{N}{\gamma} + \frac{N}{\gamma_h})||g||_o + NK(1 + \frac{\sqrt{n}}{\gamma})||\underline{q}||_{1/2,\Gamma} \right.$$

$$\left. + N(1 + \frac{\sqrt{n}}{\gamma_h})||\underline{q}_*^h||_1 + \frac{NB}{\hat{\beta}} + \frac{NB_h}{\hat{\beta}_h} \right) \left(1 + \frac{\sqrt{n}}{\gamma_h} \right)$$

and

$$C_2 = \sqrt{n}/\hat{\xi}.$$

We now estimate the error of the pressure approximation. From (3.19) we have that

$$b(\underline{v}^h, \hat{p}^h - p^h) = b(\underline{v}^h, \hat{p}^h - p) - a_o(\underline{u} - \underline{u}^h, \underline{v}^h)$$

$$- a_1(\underline{u} - \underline{u}^h, \underline{u}, \underline{v}^h) - a_1(\underline{u}^h, \underline{u} - \underline{u}^h, \underline{v}^h) \qquad \forall\ \underline{v}^h \in \underline{V}_o^h$$

where $\hat{p}^h$ is arbitrary in S_0^h. Letting $\underline{v}^h = \underline{w}^h \varepsilon \underline{W}^h$, we have, using (2.20), (2.24) and (2.28),

$$b\left(\frac{\underline{w}^h}{||\underline{w}^h||_1}, \hat{p}^h - p^h\right) \leq \sqrt{n}\, ||\hat{p}^h - p||_0 + (\nu + M||g||_0 + N||\underline{u}||_1 + N||\underline{u}^h||_1)||\underline{u}^h - \underline{u}||_1,$$

or, taking the supremum over $\underline{w}^h \varepsilon \underline{W}^h$ and using (3.7),

$$||\hat{p}^h - p^h||_0 \leq \frac{1}{\hat{\gamma}_h}\left(\sqrt{n}\, ||\hat{p}^h - p||_0 + (\nu + M||g||_0 + N||\underline{u}||_1 + N||\underline{u}^h||_1)||\underline{u}^h - \underline{u}||_1\right).$$

Then, using the triangle inequality, (2.53), (3.10) and (3.15), and taking the infimum over $\hat{p}^h \varepsilon S_0^h$, yield that (3.16) holds with

$$C_3 = \frac{C_1}{\hat{\gamma}_h}\left\{\nu + (M + \frac{N}{\gamma_h} + \frac{N}{\gamma})||g||_0 + N(1 + \frac{\sqrt{n}}{\gamma_h})||\underline{g}_*^h||_1 + NK(1 + \frac{\sqrt{n}}{\gamma})||\underline{g}||_{1/2,\Gamma}\right.$$

$$\left. + \frac{NB_h}{\hat{\beta}_h} + \frac{NB}{\hat{\beta}}\right\}$$

and

$$C_4 = 1 + \frac{\sqrt{n}}{\hat{\gamma}_h} + \frac{C_3}{C_1} C_2 \Theta. \qquad \blacksquare$$

The condition needed to prove the uniqueness of $\underline{u}$, i.e. $\xi > 0$, was also needed in the proof of the error estimates. Also, note that if $\underline{V}^h$ is chosen so that $\underline{Z}^h \subset \underline{Z}^h$, so that $\Theta = 0$, then the velocity error $||\underline{u} - \underline{u}^h||_1$ uncouples from the pressure error, i.e. (3.15) is replaced by

$$||\underline{u} - \underline{u}^h||_1 \leq C_1 \inf_{\hat{\underline{u}}^h \varepsilon \hat{\underline{V}}^h} ||\underline{u} - \hat{\underline{u}}^h||_1 \quad \text{if} \quad \underline{Z}^h \subset \underline{Z}.$$

In addition, we note that if $\underline{q}_*$ and $\underline{q}_*^h$ are chosen to be divergence and discretely divergence free, respectively, i.e.

$$b(\underline{q}_*,\psi) = 0 \quad \forall\ \psi \in L_0^2(\Omega) \quad \text{and} \quad b(q_*^h,\psi^h) = 0 \quad \forall\ \psi^h \in S_0^h,$$

then the terms involving $\sqrt{n}\ ||\underline{q}||_{1/2,\Gamma}$ and $\sqrt{n}\ ||\underline{q}_*||_1$ may be omitted from the definitions of C_i, $i = 1,\ldots,4$.

It is easily seen, by examining the constants C_i, $i = 1,..,4$, that as $\hat{\xi} \to 0$, these constants become arbitrarily large. Examining $\hat{\xi}$, which is defined by (2.55), it is easily shown that, as a function of ν,

$$\hat{\xi} = (\nu-K_1) - (K_2\nu+K_3)/(\nu-K_1)$$

where K_i, $i = 1,\ldots,3$, are non-negative constants independent of ν. Then, since

$$\frac{d\hat{\xi}}{d\nu} = 1 + \frac{K_3+K_1K_2}{(\nu-K_1)^2} > 0,$$

we see that $\hat{\xi}$ decreases with ν. Note that for $\hat{\xi} > 0$, we must have $\nu > K_1$. We recall that for small $\hat{\xi}$, $C_i = O(1/\hat{\xi})$. If ν^* denotes the critical value of ν for which $\hat{\xi} = 0$, it can be shown that, for $\nu > \nu^*$

$$C_i = O\left(\frac{1}{\nu-\nu^*}\right) \quad \text{as} \quad \nu \to \nu^*,$$

i.e. the constants in the error estimates blow up algebraically as ν approaches the critical value ν^*. In particular, it is important to note that the C_i's do not blow up exponentially in $1/(\nu-\nu^*)$, and, on the other hand, they do not blow up as $1/(\nu-\nu^*)^{1/2}$ either.

3.3 - Iterative Methods for the Discrete Equations

In this section we examine three iterative methods for the solution of the discrete equations (3.1)-(3.3). Here we follow closely [4] and especially [5]. Throughout this section we will denote by $(\underline{u}^h,p^h)$ the exact solution of the discrete equations, and by $\{\underline{u}_j,p_j\}$, $j = 0,1,2,\ldots$, the sequence of approximations to $(\underline{u}^h,p^h)$ defined by the iterative method.

We first consider Newton's method. Given $\underline{u}_0$ such that $\underline{u}_0\big|_\Gamma = \underline{q}^h$, the sequence $\{\underline{u}_j,p_j\}$, $j \geq 1$, is defined by: seek $\underline{u}_j \in \hat{\underline{V}}^h$, $p_j \in S_0^h$ such that

$$a_0(\underline{u}_j,\underline{v}^h) + a_1(\underline{u}_j,\underline{u}_{j-1},\underline{v}^h) + a_1(\underline{u}_{j-1},\underline{u}_j,\underline{v}^h) + b(\underline{v},p_j)$$

$$= \langle f,\underline{v}^h\rangle + a_1(\underline{u}_{j-1},\underline{u}_{j-1},\underline{v}^h) \quad \forall\ \underline{v}^h \in \underline{V}_0^h, \tag{3.22}$$

$$b(\underline{u}_j,\psi^h) = -\langle g,\psi^h\rangle \quad \forall\ \psi^h \in S_0^h. \tag{3.23}$$

Note that we require $\underline{u}_0$ to satisfy the boundary conditions, but we do not require that

$$b(\underline{u}_0,\psi^h) = -\langle g,\psi^h\rangle \quad \forall\ \psi^h \in S_0^h. \tag{3.24}$$

Furthermore, no initial pressure p_0 is required. The following series of propositions shows that if $\underline{u}_0$ is sufficiently close to $\underline{u}^h$, then the iterates $\{\underline{u}_j,p_j\}$ are uniquely defined by (3.22) and (3.23), and that these iterates converge quadratically to $(\underline{u}^h,p^h)$.

PROPOSITION 3.4 - *Let* $d \equiv \hat{\xi}_h/2N$ *and suppose that* $||\underline{u}_0-\underline{u}^h||_1 \leq d$ *and* $\underline{u}_0\big|_\Gamma = \underline{q}^h$. *Then the sequence of iterates* $\{u_j,p_j\}$, $j \geq 1$, *are uniquely defined by* (3.22) *and* (3.23) *and* $||\underline{u}_j-\underline{u}^h||_1 \leq d$, $j \geq 1$.

Proof: We show that the results are true for j when they are true for $j-1$. By hypothesis, they are true for $j = 0$. To show that $(\underline{u}_j,p_j)$ is uniquely defined, we need only show that the finite dimensional system (3.22) and (3.23) possesses only the trivial solution when the right hand sides vanish, i.e. that the problem of finding $\underline{\tilde{u}} \in \underline{V}_0^h$ and $\tilde{p} \in S_0^h$ such that

$$a_0(\underline{\tilde{u}},\underline{v}^h) + a_1(\underline{\tilde{u}},\underline{u}_{j-1},\underline{v}^h) + a_1(\underline{u}_{j-1},\underline{\tilde{u}},\underline{v}^h) + b(\underline{v}^h,\tilde{p}) = 0 \quad \forall\ \underline{v}^h \in \underline{V}_0^h, \tag{3.25}$$

$$b(\underline{\tilde{u}},\psi^h) = 0 \quad \forall\ \psi^h \in S_0^h \tag{3.26}$$

has only the solution $\tilde{u} = 0$ and $\tilde{p} = 0$. We set $\underline{v}^h = \tilde{u}$ in (3.25) and $\psi^h = p^h$ in (3.26). Then $b(\underline{\tilde{u}},p^h) = 0$ and

$$a_0(\underline{\tilde{u}},\underline{\tilde{u}}) + a_1(\underline{\tilde{u}},\underline{u}_{j-1},\underline{\tilde{u}}) + a_1(\underline{u}_{j-1},\underline{\tilde{u}},\underline{\tilde{u}}) = 0.$$

Then, using (2.12), (2.24) and (2.27),

$$(\nu-M||g||_0-N||\underline{u}_{j-1}||_1)||\underline{\tilde{u}}||_1^2 \leq 0$$

or

$$(\nu-M||g||_0-N||\underline{u}^h||_1-N||\underline{u}^h-\underline{u}_{j-1}||_1)||\underline{\tilde{u}}||_1^2 \leq 0,$$

or, comparing with (3.13),

$$(\hat{\xi}_h - N||\underline{u}^h - \underline{u}_{j-1}||_1)||\tilde{\underline{u}}||_1^2 \leq 0.$$

But, by hypothesis, $||\underline{u}^h - \underline{u}_{j-1}||_1 \leq \hat{\xi}_h/2N$, so that

$$\frac{1}{2}\hat{\xi}_h||\tilde{\underline{u}}||_1^2 \leq 0$$

and, since we are assuming that $\hat{\xi}_h > 0$, i.e. that the discrete solution $(\underline{u}^h, p^h)$ is uniquely determined, we have $\tilde{\underline{u}} = 0$. Now, with $\tilde{\underline{u}} = 0$, (3.25) yields

$$b(\underline{v}^h, \tilde{p}) = 0 \quad \forall\ \underline{v}^h \in \underline{V}_0^h$$

and, in particular, for all $\underline{w}^h \in \underline{W}^h$. Then, using (3.7), we easily have that $\tilde{p} = 0$.

We now show that $||\underline{u}_j - \underline{u}^h|| \leq d$. Subtracting (3.1) from (3.22) and (3.2) from (3.23)

$$a_0(\underline{u}_j - \underline{u}^h, \underline{v}^h) + b(\underline{v}^h, p_j - p^h) = a_1(\underline{u}^h, \underline{u}^h, \underline{v}^h) + a_1(\underline{u}_{j-1}, \underline{u}_{j-1}, \underline{v}^h)$$

$$- a_1(\underline{u}_j, \underline{u}_{j-1}, \underline{v}^h) - a_1(\underline{u}_{j-1}, \underline{u}_j, \underline{v}^h) \quad \forall\ \underline{v}^h \in \underline{V}_0^h, \tag{3.27}$$

$$b(\underline{u}_j - \underline{u}^h, \psi^h) = 0 \quad \forall\ \psi^h \in S_0^h. \tag{3.28}$$

Letting $\underline{v}^h = \underline{u}_j - \underline{u}^h \in \underline{V}_0^h$ and $\psi^h = p_j - p^h \in S_0^h$ in (3.27) and (3.28), respectively, and then combining yields, using (2.12),

$$a_0(\underline{u}_j-\underline{u}^h,\underline{u}_j-\underline{u}^h) + a_1(\underline{u}_j-\underline{u}^h,\underline{u}_{j-1},\underline{u}_j-\underline{u}^h) = a_1(\underline{u}_{j-1}-\underline{u}^h,\underline{u}_{j-1}-\underline{u}^h,\underline{u}_j-\underline{u}^h).$$

Then, using (2.24) and (2.27),

$$(\nu-M||g||_0-N||\underline{u}_{j-1}||_1)||\underline{u}_j-\underline{u}^h||_1 \leq N||\underline{u}_{j-1}-\underline{u}^h||_1^2$$

or, by the triangle inequality and (3.13),

$$(\hat{\xi}_h-N||\underline{u}_{j-1}-\underline{u}^h||_1)||\underline{u}_j-\underline{u}^h||_1 \leq N||\underline{u}_{j-1}-\underline{u}^h||_1^2.$$

But, by hypothesis, $||\underline{u}_{j-1}-\underline{u}^h||_1 \leq d = \hat{\xi}/2N$. Therefore

$$||\underline{u}_j-\underline{u}^h||_1 \leq \frac{1}{d}\,||\underline{u}_{j-1}-\underline{u}^h||_1^2 \tag{3.29}$$

or, $||\underline{u}_j-\underline{u}^h||_1 \leq d$. ∎

PROPOSITION 3.5 - *If* $||\underline{u}_0-\underline{u}^h||_1 \leq d$, *then for* $j \geq 1$

$$||\underline{u}_j-\underline{u}^h||_1 \leq ||\underline{u}_0-\underline{u}^h||_1^2/d^{2j-1}. \tag{3.30}$$

Further, if $||\underline{u}_0-\underline{u}^h||_1 = d\varepsilon$ *with* $0 < \varepsilon < 1$, *then for* $j \geq 1$,

$$||\underline{u}_j-\underline{u}^h||_1 \leq d\varepsilon^{2^j} \tag{3.31}$$

and for some constant $C > 0$,

$$||p_j-p^h||_0 \leq C\varepsilon^{2^j}. \tag{3.32}$$

Proof: (3.30) and (3.31) follow from (3.29). Subtracting (3.1) from (3.22) and restricting $\underline{v}^h = \underline{w}^h \in \underline{W}^h$, yields

$$b(\underline{w}^h,p_j-p^h) = -a_o(\underline{u}_j-\underline{u}^h,\underline{w}^h) + a_1(\underline{u}_{j-1}-\underline{u}^h,\underline{u}_{j-1}-\underline{u}^h,\underline{w}^h)$$

$$- a_1(\underline{u}_{j-1},\underline{u}_j-\underline{u}^h,\underline{w}^h) - a_1(\underline{u}_j-\underline{u}^h,\underline{u}_{j-1},\underline{w}^h) \quad \forall\ \underline{w}^h \in \underline{W}^h.$$

Then, using (2.24), (2.28) and (3.7),

$$\hat{\gamma}_h||p_j-p^h||_o \leq (\nu+M||g||_o)||\underline{u}_j-\underline{u}^h||_1+N\Big[||\underline{u}_{j-1}-\underline{u}^h||_1^2$$

$$+ 2||\underline{u}_{j-1}-\underline{u}^h||_1\ ||\underline{u}_j-\underline{u}^h||_1 + 2||\underline{u}^h||\ ||\underline{u}_j-\underline{u}^h||_1\Big],$$

or, using (3.31),

$$||p_j-p^h||_o \leq \frac{N}{\hat{\gamma}_h}\left[(d\varepsilon^{2^{j-1}})^2 + 2d^2\varepsilon^{2^{j-1}}\varepsilon^{2^j} + 2||\underline{u}^h||_1\, d\varepsilon^{2^j}\right]$$

$$+ \frac{1}{\hat{\gamma}_h}(\nu+M||g||_o)d\varepsilon^{2^j} \leq C\varepsilon^{2^j}.$$

■

We next consider the modified Newton, or chord method. Given $\underline{u}_o$ such that $\underline{u}_o\big|_\Gamma = \underline{g}^h$, the sequence $\{\underline{u}_j,p_j\}$, $j \geq 1$, is defined by: seek $\underline{u}_j \in \hat{\underline{V}}^h$ $p_j \in S_o^h$ such that

$$a_o(\underline{u}_j,\underline{v}^h) + a_1(\underline{u}_j,\underline{u}_o,\underline{v}^h) + a_1(\underline{u}_o,\underline{u}_j,\underline{v}^h) + b(\underline{v}^h,p_j)$$

$$= \langle \underline{f},\underline{v}^h\rangle + a_1(\underline{u}_{j-1},\underline{u}_o,\underline{v}^h) + a_1(\underline{u}_o,\underline{u}_{j-1},\underline{v}^h) - a_1(\underline{u}_{j-1},\underline{u}_{j-1},\underline{v}^h) \quad \forall\ \underline{v}^h \in \underline{V}_o^h$$

(3.33)

$$b(\underline{u}_j,\psi^h) = -\langle g,\psi^h\rangle \quad \forall \ \psi^h \in S_0^h. \tag{3.34}$$

Once again we require only that $\underline{u}_0$ satisfy the boundary condition. We do not require an initial pressure p_0 or for $\underline{u}_0$ to satisfy (3.24). The advantage of this method, with respect to Newton's method or the simple iterative method scheme defined below, is that the linear system to be solved at each iteration always involves the same coefficient matrix. On the other hand, the chord method is only locally linearly convergent. In a manner entirely analogous to that for Newton's method, we can prove the following result.

PROPOSITION 3.6 - *Let* $d \equiv \hat{\xi}_h/2N$ *and suppose that* $||\underline{u}_0-\underline{u}^h||_1 \leq d/2$ *and* $\underline{u}_0\big|_\Gamma = \underline{g}^h$. *Then the sequence of iterates* $\{\underline{u}_j,p_j\}$, $j \geq 1$, *are uniquely defined by (3.33) and (3.34) and* $||\underline{u}_j-\underline{u}^h||_1 \leq d/2$, $j \geq 1$. *Moreover, if* $||\underline{u}_0-\underline{u}^h||_1 = d\varepsilon/2$ *with* $0 < \varepsilon < 1$, *then for* $j \geq 1$,

$$||\underline{u}_j-\underline{u}^h||_1 \leq d\varepsilon^{j+1}/2$$

and

$$||p_j-p^h||_0 \leq C\varepsilon^{j+1}.$$

Following the discussion at the end of Section 3.2, we note that the attraction balls for the Newton and chord methods, whose radii are proportional to $\hat{\xi}_h$, vanish as ν approaches ν^*. Indeed, then radii are $O(\nu-\nu^*)$ as ν approaches ν^*.

The final scheme we consider is the following simple iteration. Suppose

$\hat{\xi}_h > 0$ so that the solution $(\underline{u}^h, p^h)$ of the discrete equations is uniquely defined. Then, given $\underline{u}_0 \in \underline{V}^h$, the sequence $\{\underline{u}_j, p_j\}$, $j \geq 1$, is defined by: seek $\underline{u}_j \in \underline{\hat{V}}^h$ and $p_j \in S_0^h$ such that

$$a_0(\underline{u}_j, \underline{v}^h) + a_1(\underline{u}_{j-1}, \underline{u}_j, \underline{v}^h) + b(\underline{v}^h, p_j) = \langle \underline{f}, \underline{v}^h \rangle \quad \forall\ \underline{v}^h \in \underline{V}_0^h, \tag{3.35}$$

$$b(\underline{u}_j, \psi^h) = -\langle g, \psi^h \rangle \quad \forall\ \psi^h \in S_0^h. \tag{3.36}$$

For this scheme, one can prove the following results.

PROPOSITION 3.7 - *Let* $\hat{\xi}_h > 0$ *so that the solution* $\{\underline{u}^h, p^h\}$ *of the discrete equations is uniquely defined. Then, given* $\underline{u}_0 \in \underline{V}^h$, *the sequence* $\{\underline{u}_j, p_j\}$, $j \geq 1$, *is uniquely defined by* (3.35) *and* (3.36). *Moreover, if* α *is a constant such that*

$$\alpha(\nu - M||g||_0) = N||\underline{u}^h||_1,$$

then $\alpha < 1$ *and, for* $j \geq 1$,

$$||\underline{u}_j - \underline{u}^h||_1 \leq \alpha^j ||\underline{u}_0 - \underline{u}^h||_1.$$

Proof: The proof follows closely those of [4] and [5] for the homogeneous stationary Navier-Stokes equations. We only note that necessarily $\alpha < 1$ because we have assumed that $\xi_h = \nu - M||g||_0 - N||\underline{u}^h||_1 > 0$. ∎

The simple iterative scheme (3.35) and (3.36) is thus linearly and globally convergent. It requires the solution of a different matrix problem for each iteration. No initial pressure p_0 is needed, nor does the initial velocity $\underline{u}_0$ need to satisfy (3.24) or the boundary condition $\underline{u}_0|_\Gamma = \underline{g}^h$. In particular, we may start with the initial condition $\underline{u}_0 \equiv \underline{0}$.

4. NUMERICAL EXAMPLES

In this section we present three numerical examples which illustrate some of the theoretical results of the previous sections. Specifically, we wish to illustrate the estimates for the errors in the approximate solution for the velocity and the quadratic convergence of the Newton iterates. An extensive report of the numerous other computational results will be made elsewhere. The first two examples are artificial, i.e. we define an exact solution and then adjust the data $\underline{f}$, g and $\underline{g}$ so that the governing equations (2.1)-(2.3) are satisfied. The third example is a physical flow, namely the plane flow in the neighborhood of a stagnation point. For all three examples the region Ω is the unit square $\{0 < x < 1, 0 < y < 1\}$.

The pair of finite element spaces which is used in the examples is defined as follows. We subdivide the region Ω into quadrilaterals and then divide each quadrilateral into two triangles by drawing a diagonal. For the velocity space $\underline{V}^h$ we choose vector valued functions whose components are continuous piecewise linear functions over the *triangles*. For the pressure space, we first let S^h consist of piecewise constant functions over the *quadrilaterals*. We then constrain the functions in S^h to have zero mean over Ω. In addition, on a regular or graded grid, a second spurious singular mode must be eliminated from S^h, while on an irregular grid, there is a spurious nonsingular mode which must be eliminated. See [10] and [12] for details. Then the space S_0^h in which we seek the approximate pressure is the space S^h with the above two constraints imposed. In actual computations, neither constraint need be explicitly imposed on S^h beforehand, but rather they are imposed *a posteriori* on the computed pressure by a simple post processing procedure. The finite element pair thus chosen has been shown to be stable, i.e. that (3.6) and (3.7) hold, on regular grids [10] and on general grids [12]. Then, according to the results of the previous section, we expect that if $\underline{u} \in \underline{H}^2(\Omega)$ and $p \in H^1(\Omega)$,

$$||\underline{u}-\underline{u}^h||_1 \leq Ch \quad \text{and} \quad ||\underline{u}-\underline{u}^h||_0 \leq Ch^2, \tag{4.1}$$

where h is a measure of the grid size.

For each example we divide the unit square Ω into smaller squares of side h. We report the $\underline{L}^2(\Omega)$ and $\underline{H}^1(\Omega)$ velocity errors on pairs of grids, and for each pair, the rate of convergence of the approximate solutions. These rates are computed by comparing the errors on two grids. Specifically, we use the formula

$$\text{rate} = \ln(\varepsilon_1/\varepsilon_2)/\ln(h_1/h_2) \tag{4.2}$$

where ε_i denotes the error on the grid h_i. In addition, we give some examples reporting on the distance, measured in the $\underline{H}^1(\Omega)$-norm, between the last Newton iterate and the preceeding Newton iterates. We note that for all the computations, we first computed a few simple iterates, usually two or three, before switching to the Newton iteration. This enabled us to start the computation with an arbitrary initial velocity vector, e.g. the zero vector.

For all three examples the exact solution $(\underline{u},p)$ was known. The computations were carried out with the choice $\nu = 1$ with $\underline{g}^h$ defined to be the interpolant of the exact solution $\underline{u}$ restricted to the boundary Γ of Ω.

The norms of the error $(\underline{u}-\underline{u}^h)$ were computed using high order quadrature rules.

The first example, for which $\underline{f} \neq 0$ and $g \neq 0$ but $\underline{q} = 0$, has the exact solution

$$\underline{u} = \begin{Bmatrix} \sin \pi x \sin 2\pi y \\ x^2(1-x)\sin \pi y \end{Bmatrix},$$

$$p = \cos \pi x(1 + y(y^2-4))$$

and illustrates the effects of an inhomogeneity in the continuity equation. The second example, for which $\underline{f} \neq 0$, $g \neq 0$ and $\underline{q} \neq 0$, has the exact solution

$$\underline{u} = \begin{Bmatrix} Ae^{\lambda y} \cos \lambda x \\ B \ln[(1+x)^2 + y^2] \end{Bmatrix},$$

$$p = C(\cos \pi x)(1-4y+y^3)$$

where B, C and λ are constants which were chosen to be .1, .1 and -1, respectively in the computations reported. A is obtained from the expression

$$A = \frac{2\lambda B}{(e^{\lambda}-1)(1-\cos \lambda)} \left(\ln(5) - \frac{5}{2} \ln(2) + \arctan(2) - \frac{\pi}{4} \right)$$

which arises from enforcing (2.4). The third example is that of plane flow in the neighborhood of a stagnation point, i.e. Heimenz flow. This flow, which is defined for $-\infty < x < \infty$ and $y > 0$, has the exact solution [16]

$$\underline{u} = \begin{Bmatrix} x\phi'(\eta) \\ -\phi(\eta) \end{Bmatrix}$$

where $\eta = y/\sqrt{\nu}$ and $\phi(\eta)$ is the solution of the boundary value problem

$$\phi''' + \phi''\phi - \phi'^2 + 1 = 0$$

$$\phi(0) = \phi'(0) = 0 \quad \text{and} \quad \phi'(\infty) = 1.$$

For this example, $\underline{f} = 0$ and $g = 0$ but $\underline{q} \neq 0$.

TABLE I

Computational Results for Example 1

Grid Size	$\underline{L}^2$-Error	$\underline{L}^2$-Rate	$\underline{H}^1$-Error	$\underline{H}^1$-Rate	$\underline{H}^1$-Distance of Newton Iterates
1/5	.4399 (-1)		.8152		.9623 (-2) .3948 (-4) .3685 (-11)
$\frac{1}{10}$	.1152 (-1)	1.933	.4090	.995	.1548 (-1) .1089 (-2) .7512 (-9) .4617 (-12)
$\frac{1}{6}$	.3107 (-1)		.6792		.1261 (-1) .4167 (-2) .3129 (-7) .1316 (-12)
$\frac{1}{12}$	.8047 (-2)	1.949	.3412	.993	.1619 (-1) .5575 (-3) .9313 (-10) .2059 (-11)
$\frac{1}{7}$	.2308 (-1)		.5829		.1326 (-1) .5662 (-4) .9191 (-11)
$\frac{1}{14}$	.5934 (-2)	1.960	.2926	.994	.1663 (-1) .3660 (-3) .9326 (-10) .2456 (-11)

TABLE II

Computational Results for Example 2

Grid Size	$\underline{L}^2$-Error	$\underline{L}^2$-Rate	$\underline{H}^1$-Error	$\underline{H}^1$-Rate	$\underline{H}^1$-Distance of Newton Iterates
1/5	.1906 (-3)		.6039 (-2)		.2580 .6363 (-3) .6454 (-9) .4083 (-13)
1/10	.5022 (-4)	1.925	.3028 (-2)	.996	.3921 .5213 (-3) .2790 (-9) .2353 (-12)
1/6	.1452 (-3)		.5032 (-2)		.2896 .6659 (-3) .7495 (-9) .9605 (-13)
1/12	.3454 (-4)	2.072	.2530 (-2)	.992	.4345 .4733 (-3) .2282 (-9) .1758 (-12)
1/7	.1010 (-3)		.4334 (-2)		.3182 .6580 (-3) .4913 (-9) .4241 (-13)
1/14	.2579 (-4)	1.969	.2158 (-2)	1.006	.4732 .2477 (-3) .1250 (-9) .3396 (-12)

TABLE III

Computational Results for Example 3

Grid Size	L^2-Error	L^2-Rate	H^1-Error	H^1-Rate	H^1-Distance of Newton Iterates
1/5	.1804 (-2)		.5001 (-1)		.1698 (-1) .3498 (-6) .1655 (-12)
1/10	.4573 (-3)	1.98	.2531 (-1)	.98	.8707 (-2) .7857 (-2) .2553 (-7) .1429 (-11)
1/6	.1259 (-2)		.4244 (-1)		.1038 (-1) .9765 (-2) .1433 (-6) .2742 (-12)
1/12	.3128 (-3)	2.01	.2111 (-1)	1.01	.7452 (-2) .8445 (-2) .5028 (-2) .6624 (-2) .2783 (-7) .7931 (-12)

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